

# ON JOINT WEAK CONVERGENCE OF PARTIAL SUM AND MAXIMA PROCESSES

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**ABSTRACT.** For a strictly stationary sequence of random variables we derive functional convergence of the joint partial sum and partial maxima processes under joint regular variation with index  $\alpha \in (0, 1)$  and weak dependence conditions. The convergence takes place in the space of  $\mathbb{R}^2$ -valued càdlàg functions on  $[0, 1]$ , with the Skorohod weak  $M_1$  topology. We also show that this topology in general can not be replaced by the stronger (standard)  $M_1$  topology.

## 1. INTRODUCTION

Consider a strictly stationary sequence of random variables  $(X_n)$  and denote by  $S_n = X_1 + \dots + X_n$  and  $M_n = \max\{X_i : i = 1, \dots, n\}$ ,  $n \geq 1$ , its accompanying sequences of partial sums and maxima, respectively. It is well known that if the  $X_n$  are i.i.d. and regularly varying with index  $\alpha \in (0, 2)$ , then

$$\frac{S_n - b'_n}{a'_n} \xrightarrow{d} S,$$

for some  $a'_n > 0$  and  $b'_n \in \mathbb{R}$  and some  $\alpha$ -stable random variable  $S$ , and

$$\frac{M_n}{a''_n} \xrightarrow{d} Y,$$

for some  $a''_n > 0$  and some random variable  $Y$  with a Fréchet distribution, see for example Gnedenko and Kolmogorov [10] and Resnick [16]. The weak convergence of partial maxima holds also for  $\alpha \geq 2$ .

The joint weak limiting behavior of  $(S_n, M_n)$  with appropriate centering and scaling was investigated by Chow and Teugels [7]. They also obtained a functional limit theorem for a suitably normalized joint partial sum and partial maxima processes. See also Anderson and Turkman [1] and Resnick [15] for related results.

In this paper, under the properties of weak dependence and joint regular variation with index  $\alpha \in (0, 1)$  for the sequence  $(X_n)$ , we investigate functional convergence of the joint partial sum and partial maxima processes  $L_n(\cdot) = (V_n(\cdot), W_n(\cdot))$  in the space  $D([0, 1], \mathbb{R}^2)$ , where

$$V_n(t) = \frac{S_{[nt]}}{a_n}, \quad W_n(t) = \frac{M_{[nt]}}{a_n}, \quad t \in [0, 1],$$

with  $(a_n)$  being a sequence of positive real numbers such that

$$nP(|X_1| > a_n) \rightarrow 1, \tag{1.1}$$

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2010 *Mathematics Subject Classification.* Primary 60F17; Secondary 60G52, 60G55, 60G70.

*Key words and phrases.* functional limit theorem, regular variation, weak  $M_1$  topology, extremal process, Lévy process.

as  $n \rightarrow \infty$ . Here,  $\lfloor x \rfloor$  represents the integer part of the real number  $x$  and  $D([0, 1], \mathbb{R}^2)$  is the space of  $\mathbb{R}^2$ -valued càdlàg functions on  $[0, 1]$ .

The main result of our article shows that for a strictly stationary, regularly varying sequence of dependent random variables  $(X_n)$  with index  $\alpha \in (0, 1)$ , for which clusters of high-threshold excesses can be broken down into asymptotically independent blocks, the stochastic processes  $L_n(\cdot)$  converge in the space  $D([0, 1], \mathbb{R}^2)$  endowed with the Skorohod weak  $M_1$  topology under the condition that all extremes within each cluster of big values have the same sign. This topology is weaker than the more commonly used Skorohod  $J_1$  topology, the latter being appropriate when there is no clustering of extremes (which for example occurs in the i.i.d. case).

The paper is organized as follows. In Section 2 we introduce the essential ingredients about regular variation, weak dependence and Skorohod topologies. In Section 3 we state and prove our main result using a new limit theorem derived recently by Basrak and Tafař [6] for the time-space point processes  $N_n = \sum_{i=1}^n \delta_{(i/n, X_i/a_n)}$ . Finally, in Section 4 we illustrate by an example that the weak  $M_1$  convergence in our main theorem, in general, can not be replaced by the standard  $M_1$  convergence.

## 2. PRELIMINARIES

**2.1. Regular variation.** Let  $\mathbb{E}^d = [-\infty, \infty]^d \setminus \{0\}$ . We equip  $\mathbb{E}^d$  with the topology in which a set  $B \subset \mathbb{E}^d$  has compact closure if and only if it is bounded away from zero, that is, if there exists  $u > 0$  such that  $B \subset \mathbb{E}_u^d = \{x \in \mathbb{E}^d : \|x\| > u\}$ . Here  $\|\cdot\|$  denotes the max-norm on  $\mathbb{R}^d$ , i.e.  $\|x\| = \max\{|x_i| : i = 1, \dots, d\}$  where  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ . Denote by  $C_K^+(\mathbb{E}^d)$  the class of all nonnegative, continuous functions on  $\mathbb{E}^d$  with compact support.

We say that a strictly stationary process  $(X_n)_{n \in \mathbb{Z}}$  is *(jointly) regularly varying* with index  $\alpha \in (0, \infty)$  if for any nonnegative integer  $k$  the  $kd$ -dimensional random vector  $X = (X_1, \dots, X_k)$  is multivariate regularly varying with index  $\alpha$ , i.e. there exists a random vector  $\Theta$  on the unit sphere  $\mathbb{S}^{kd-1} = \{x \in \mathbb{R}^{kd} : \|x\| = 1\}$  such that for every  $u \in (0, \infty)$  and as  $x \rightarrow \infty$ ,

$$\frac{\mathbb{P}(\|X\| > ux, X/\|X\| \in \cdot)}{\mathbb{P}(\|X\| > x)} \xrightarrow{w} u^{-\alpha} \mathbb{P}(\Theta \in \cdot), \quad (2.1)$$

the arrow “ $\xrightarrow{w}$ ” denoting weak convergence of finite measures.

Regular variation can be expressed in terms of vague convergence of measures on  $\mathbb{E}$  as follows: for  $a_n$  as in (1.1),

$$n\mathbb{P}(a_n^{-1}X_i \in \cdot) \xrightarrow{v} \mu(\cdot), \quad (2.2)$$

where the limit  $\mu$  is a nonzero Radon measure on  $\mathbb{E}$  that satisfies  $\mu(\mathbb{E} \setminus \mathbb{R}) = 0$ .

Theorem 2.1 in Basrak and Segers [5] provides a convenient characterization of joint regular variation: it is necessary and sufficient that there exists a process  $(Y_n)_{n \in \mathbb{Z}}$  with  $\mathbb{P}(|Y_0| > y) = y^{-\alpha}$  for  $y \geq 1$  such that as  $x \rightarrow \infty$ ,

$$((x^{-1} X_n)_{n \in \mathbb{Z}} \mid |X_0| > x) \xrightarrow{\text{fidi}} (Y_n)_{n \in \mathbb{Z}}, \quad (2.3)$$

where “ $\xrightarrow{\text{fidi}}$ ” denotes convergence of finite-dimensional distributions. The process  $(Y_n)$  is called the *tail process* of  $(X_n)$ .

**2.2. Point processes and dependence conditions.** Let  $(X_n)$  be a strictly stationary sequence of random variables and assume it is jointly regularly varying with index  $\alpha > 0$ . Let  $(Y_n)$  be the tail process of  $(X_n)$ . In order to obtain weak convergence of the processes  $L_n(\cdot)$  we will use the so-called complete convergence result for the corresponding point processes of jumps obtained recently by Basrak and Taïro [6], and then by the continuous mapping theorem and some properties of Skorohod topologies we will transfer this convergence result to the joint partial sum and maxima processes.

Let

$$N_n = \sum_{i=1}^n \delta_{(i/n, X_i/a_n)} \quad \text{for all } n \in \mathbb{N},$$

with  $a_n$  as in (1.1). The point process convergence for the sequence  $(N_n)$  was already established by Basrak et al. [4] on the space  $[0, 1] \times \mathbb{E}_u$  for any threshold  $u > 0$ , with the limit depending on that threshold. Recently Basrak and Taïro [6] obtained a new convergence result for  $N_n$  without the restriction to various domains (i.e. their convergence result holds on the space  $[0, 1] \times \mathbb{E}$ ).

The appropriate weak dependence conditions for this convergence result are given below. With them we will be able to control the dependence in the sequence  $(X_n)$ .

**Condition 2.1.** There exists a sequence of positive integers  $(r_n)$  such that  $r_n \rightarrow \infty$  and  $r_n/n \rightarrow 0$  as  $n \rightarrow \infty$  and such that for every  $f \in C_K^+([0, 1] \times \mathbb{E})$ , denoting  $k_n = \lfloor n/r_n \rfloor$ , as  $n \rightarrow \infty$ ,

$$\mathbb{E} \left[ \exp \left\{ - \sum_{i=1}^n f \left( \frac{i}{n}, \frac{X_i}{a_n} \right) \right\} \right] - \prod_{k=1}^{k_n} \mathbb{E} \left[ \exp \left\{ - \sum_{i=1}^{r_n} f \left( \frac{kr_n}{n}, \frac{X_i}{a_n} \right) \right\} \right] \rightarrow 0. \quad (2.4)$$

It can be shown that Condition 2.1 is implied by the strong mixing property (cf. Krizmanić [12]). Condition 2.1 is slightly stronger than the condition  $\mathcal{A}(a_n)$  introduced by Davis and Mikosch [9].

**Condition 2.2.** There exists a sequence of positive integers  $(r_n)$  such that  $r_n \rightarrow \infty$  and  $r_n/n \rightarrow 0$  as  $n \rightarrow \infty$  and such that for every  $u > 0$ ,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \max_{m \leq |i| \leq r_n} |X_i| > ua_n \mid |X_0| > ua_n \right) = 0. \quad (2.5)$$

By Proposition 4.2 in Basrak and Segers [5], under Condition 2.2 the following holds

$$\theta = \mathbb{P}(\sup_{i \geq 1} \|Y_i\| \leq 1) = \mathbb{P}(\sup_{i \leq -1} \|Y_i\| \leq 1) > 0, \quad (2.6)$$

and  $\theta$  is the extremal index of the univariate sequence  $(|X_n|)$ . For a detailed discussion on joint regular variation and dependence Conditions 2.1 and 2.2 we refer to Basrak et al. [4], Section 3.4.

Under joint regular variation and Conditions 2.1 and 2.2, by Theorem 3.1 in Basrak and Taïro [6], as  $n \rightarrow \infty$ ,

$$N_n \xrightarrow{d} N = \sum_i \sum_j \delta_{(T_i, P_i \eta_{ij})} \quad (2.7)$$

in  $[0, 1] \times \mathbb{E}$ , where  $\sum_{i=1}^{\infty} \delta_{(T_i, P_i)}$  is a Poisson process on  $[0, 1] \times (0, \infty)$  with intensity measure  $Leb \times \nu$  where  $\nu(dx) = \theta \alpha x^{-\alpha-1} 1_{(0, \infty)}(x) dx$ , and  $(\sum_{j=1}^{\infty} \delta_{\eta_{ij}})_i$  is an i.i.d. sequence of point processes in  $\mathbb{E}$  independent of  $\sum_i \delta_{(T_i, P_i)}$  and with

common distribution equal to the distribution of  $\sum_j \delta_{Q_j}$ , where  $Q_j = Z_j/L_Z$ ,  $L_Z = \sup_{j \in \mathbb{Z}} |Z_j|$  and  $\sum_j \delta_{Z_j}$  is distributed as  $(\sum_{j \in \mathbb{Z}} \delta_{Y_j} \mid \sup_{i \leq -1} |Y_i| \leq 1)$ .

**2.3. The weak and strong  $M_1$  topologies.** The stochastic processes that we consider have discontinuities, and hence for the function space of sample paths of these stochastic processes we take the space  $D([0, 1], \mathbb{R}^2)$  of all right-continuous  $\mathbb{R}^2$ -valued functions on  $[0, 1]$  with left limits.

The stochastic processes  $V_n(\cdot)$  and  $W_n(\cdot)$  converge (separately) in the space  $D([0, 1], \mathbb{R})$  equipped with the standard  $M_1$  topology, see Basrak et al. [4] and Krizmanić [11]. In this paper we use the weak  $M_1$  topology, since as we show later the functional convergence for  $L_n(\cdot)$  in general fails to hold in the standard  $M_1$  topology on  $D([0, 1], \mathbb{R}^2)$ . In the sequel we give the definitions of the weak and standard (strong)  $M_1$  topologies.

For  $x \in D([0, 1], \mathbb{R}^k)$  the completed graph of  $x$  is the set

$$G_x = \{(t, z) \in [0, 1] \times \mathbb{R}^2 : z \in [[x(t-), x(t)]]\},$$

where  $x(t-)$  is the left limit of  $x$  at  $t$  and  $[[a, b]]$  is the product segment, i.e.  $[[a, b]] = [a_1, b_1] \times [a_2, b_2]$  for  $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ . We define an order on the graph  $G_x$  by saying that  $(t_1, z_1) \leq (t_2, z_2)$  if either (i)  $t_1 < t_2$  or (ii)  $t_1 = t_2$  and  $|x_j(t_1-) - z_{1j}| \leq |x_j(t_2-) - z_{2j}|$  for all  $j = 1, 2$ . Note that the relation  $\leq$  induces only a partial order on the graph  $G_x$ . A weak parametric representation of the graph  $G_x$  is a continuous nondecreasing function  $(r, u)$  mapping  $[0, 1]$  into  $G_x$ , with  $r \in C([0, 1], [0, 1])$  being the time component and  $u \in C([0, 1], \mathbb{R}^2)$  being the spatial component, such that  $r(0) = 0, r(1) = 1$  and  $u(1) = x(1)$ . Let  $\Pi_w(x)$  denote the set of weak parametric representations of the graph  $G_x$ . For  $x_1, x_2 \in D([0, 1], \mathbb{R}^2)$  define

$$d_w(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0, 1]} \vee \|u_1 - u_2\|_{[0, 1]} : (r_i, u_i) \in \Pi_w(x_i), i = 1, 2\},$$

where  $\|x\|_{[0, 1]} = \sup\{\|x(t)\| : t \in [0, 1]\}$ . Now we say that  $x_n \rightarrow x$  in  $D([0, 1], \mathbb{R}^2)$  for a sequence  $(x_n)$  in the weak Skorohod  $M_1$  (or shortly  $WM_1$ ) topology if  $d_w(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ .

Now we recall the definition of the standard  $M_1$  topology. For  $x \in D([0, 1], \mathbb{R}^2)$  let

$$\Gamma_x = \{(t, z) \in [0, 1] \times \mathbb{R}^2 : z \in [x(t-), x(t)]\},$$

where  $[a, b] = \{\lambda a + (1 - \lambda)b : 0 \leq \lambda \leq 1\}$  for  $a, b \in \mathbb{R}^2$ . We say  $(r, u)$  is a parametric representation of  $\Gamma_x$  if it is a continuous nondecreasing function mapping  $[0, 1]$  onto  $\Gamma_x$ . Denote by  $\Pi(x)$  the set of all parametric representations of the graph  $\Gamma_x$ . Then for  $x_1, x_2 \in D([0, 1], \mathbb{R}^2)$

$$d_{M_1}(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0, 1]} \vee \|u_1 - u_2\|_{[0, 1]} : (r_i, u_i) \in \Pi(x_i), i = 1, 2\}.$$

$d_{M_1}$  is a metric on  $D([0, 1], \mathbb{R}^2)$ , and the induced topology is called the (standard or strong) Skorohod  $M_1$  topology. The  $WM_1$  topology is weaker than the standard  $M_1$  topology on  $D([0, 1], \mathbb{R}^2)$ . The  $WM_1$  topology coincides with the topology induced by the metric

$$d_p(x_1, x_2) = \max\{d_{M_1}(x_{1j}, x_{2j}) : j = 1, 2\} \quad (2.8)$$

for  $x_i = (x_{i1}, x_{i2}) \in D([0, 1], \mathbb{R}^2)$  and  $i = 1, 2$ . The metric  $d_p$  induces the product topology on  $D([0, 1], \mathbb{R}^2)$ . For detailed discussion of the strong and weak  $M_1$  topologies we refer to Whitt [20], sections 12.3–12.5.

3. FUNCTIONAL CONVERGENCE OF  $L_n(\cdot)$ 

In this section we show the convergence of the joint partial sum and maxima process

$$L_n(t) = (V_n(t), W_n(t)), \quad t \in [0, 1],$$

in the space  $D([0, 1], \mathbb{R}^2)$  equipped with Skorohod weak  $M_1$  topology. We identify the limit as  $(V(\cdot), W(\cdot))$ , where  $V(\cdot)$  is a stable Lévy process and  $W(\cdot)$  an extremal process. We first represent  $L_n(\cdot)$  as the image of the time-space point process  $N_n$  under a certain sum-maximum functional. Then, using certain continuity properties of this functional, by the continuous mapping theorem we transfer the weak convergence of  $N_n$  in (2.7) to weak convergence of  $L_n(\cdot)$ .

**3.1. The sum-maximum functional.** Fix  $0 < u < \infty$  and define the sum-maximum functional

$$\Phi^{(u)}: \mathbf{M}_p([0, 1] \times \mathbb{E}) \rightarrow D([0, 1], \mathbb{R}^2)$$

by

$$\Phi^{(u)}\left(\sum_i \delta_{(t_i, x_i)}\right)(t) = \left(\sum_{t_i \leq t} x_i 1_{\{u < |x_i| < \infty\}}, \bigvee_{t_i \leq t} x_i \vee 0\right), \quad t \in [0, 1].$$

The space  $\mathbf{M}_p([0, 1] \times \mathbb{E})$  of Radon point measures on  $[0, 1] \times \mathbb{E}$  is equipped with the vague topology and  $D([0, 1], \mathbb{R}^2)$  is equipped with the weak  $M_1$  topology. For convenience we set  $\sup \emptyset = 0$ . Let  $\Lambda = \Lambda_1 \cap \Lambda_2$ , where

$$\Lambda_1 = \{\eta \in \mathbf{M}_p([0, 1] \times \mathbb{E}) : \eta(\{0, 1\} \times \mathbb{E}) = 0 = \eta([0, 1] \times \{\pm\infty, \pm u\})\},$$

$$\Lambda_2 = \{\eta \in \mathbf{M}_p([0, 1] \times \mathbb{E}) : \eta(\{t\} \times (u, \infty]) \cdot \eta(\{t\} \times [-\infty, -u)) = 0$$

for all  $t \in [0, 1]\}$ .

Observe that the elements of  $\Lambda_2$  have the property that atoms in  $[0, 1] \times \mathbb{E}_u$  with the same time coordinate are all on the same side of the time axis. Similar to Lemma 3.1 in Basrak et al. [4] one can prove the following result.

**Lemma 3.1.** *Assume that with probability one, the tail process  $(Y_i)_{i \in \mathbb{Z}}$  in (2.3) has no two values of the opposite sign. Then  $P(N \in \Lambda) = 1$ .*

Now we will show that  $\phi^{(u)}$  is continuous on the set  $\Lambda$ .

**Lemma 3.2.** *The sum-maximum functional  $\Phi^{(u)}: \mathbf{M}_p([0, 1] \times \mathbb{E}) \rightarrow D([0, 1], \mathbb{R}^2)$  is continuous on the set  $\Lambda$ , when  $D([0, 1], \mathbb{R}^2)$  is endowed with the weak  $M_1$  topology.*

*Proof.* Take an arbitrary  $\eta \in \Lambda$  and suppose that  $\eta_n \xrightarrow{v} \eta$  in  $\mathbf{M}_p([0, 1] \times \mathbb{E})$ . We need to show that  $\Phi^{(u)}(\eta_n) \rightarrow \Phi^{(u)}(\eta)$  in  $D([0, 1], \mathbb{R}^2)$  according to the  $WM_1$  topology. By Theorem 12.5.2 in Whitt [20], it suffices to prove that, as  $n \rightarrow \infty$ ,

$$d_p(\Phi^{(u)}(\eta_n), \Phi^{(u)}(\eta)) = \max_{k=1,2} d_{M_1}(\Phi_k^{(u)}(\eta_n), \Phi_k^{(u)}(\eta)) \rightarrow 0.$$

Now one can follow, with small modifications, the lines in the proof of Lemma 3.2 in Basrak et al. [4] to obtain  $d_{M_1}(\Phi_1^{(u)}(\eta_n), \Phi_1^{(u)}(\eta)) \rightarrow 0$  as  $n \rightarrow \infty$ .

Let

$$T = \{t \in [0, 1] : \eta(\{t\} \times \mathbb{E}) = 0\}.$$

Since  $\eta$  is a Radon point measure, the set  $T$  is dense in  $[0, 1]$ . Fix  $t \in T$  and take  $\epsilon > 0$  such that  $\eta([0, t] \times \{\epsilon\}) = 0$ . Later, when  $\epsilon \downarrow 0$ , we assume convergence to 0

is through a sequence of values  $(\epsilon_j)$  such that  $\eta([0, t] \times \{\epsilon_j\}) = 0$  for all  $j \in \mathbb{N}$  (this can be arranged since  $\eta$  is a Radon point measure). Since the set  $[0, t] \times [\epsilon, \infty]$  is relatively compact in  $[0, 1] \times \mathbb{E}$ , there exists a nonnegative integer  $k = k(\eta)$  such that

$$\eta([0, t] \times [\epsilon, \infty]) = k < \infty.$$

By assumption,  $\eta$  does not have any atoms on the border of the set  $[0, t] \times [\epsilon, \infty]$ . Hence, by Lemma 7.1 in Resnick [16], there exists a positive integer  $n_0$  such that for all  $n \geq n_0$  it holds that

$$\eta_n([0, t] \times [\epsilon, \infty]) = k.$$

Let  $(t_i, x_i)$  for  $i = 1, \dots, k$  be the atoms of  $\eta$  in  $[0, t] \times [\epsilon, \infty]$ . By the same lemma, the  $k$  atoms  $(t_i^{(n)}, x_i^{(n)})$  of  $\eta_n$  in  $[0, t] \times [\epsilon, \infty]$  (for  $n \geq n_0$ ) can be labeled in such a way that for every  $i \in \{1, \dots, k\}$  we have

$$(t_i^{(n)}, x_i^{(n)}) \rightarrow (t_i, x_i) \quad \text{as } n \rightarrow \infty.$$

In particular, for any  $\delta > 0$  we can find a positive integer  $n_\delta \geq n_0$  such that for all  $n \geq n_\delta$ ,

$$|t_i^{(n)} - t_i| < \delta \quad \text{and} \quad |x_i^{(n)} - x_i| < \delta \quad \text{for } i = 1, \dots, k.$$

If  $k = 0$ , then (for large  $n$ ) the atoms of  $\eta$  and  $\eta_n$  in  $[0, t] \times \mathbb{E}$  are all situated in  $[0, t] \times [-\infty, \epsilon)$ . Hence  $\Phi_2^{(u)}(\eta)(t) \in [0, \epsilon)$  and  $\Phi_2^{(u)}(\eta_n)(t) \in [0, \epsilon)$ , which imply

$$|\Phi_2^{(u)}(\eta_n)(t) - \Phi_2^{(u)}(\eta)(t)| < \epsilon. \quad (3.1)$$

If  $k \geq 1$ , take  $\delta = \epsilon$ . Then we have

$$|\Phi_2^{(u)}(\eta_n)(t) - \Phi_2^{(u)}(\eta)(t)| = \left| \bigvee_{i=1}^k x_i^{(n)} - \bigvee_{i=1}^k x_i \right| \leq \bigvee_{i=1}^k |x_i^{(n)} - x_i| < \epsilon. \quad (3.2)$$

Therefore from (3.1) and (3.2) we obtain

$$\lim_{n \rightarrow \infty} |\Phi_2^{(u)}(\eta_n)(t) - \Phi_2^{(u)}(\eta)(t)| < \epsilon,$$

and if we let  $\epsilon \rightarrow 0$ , it follows that  $\Phi_2^{(u)}(\eta_n)(t) \rightarrow \Phi_2^{(u)}(\eta)(t)$  as  $n \rightarrow \infty$ . Note that  $\Phi_2^{(u)}(\eta)$  and  $\Phi_2^{(u)}(\eta_n)$  are nondecreasing functions. Since, by Corollary 12.5.1 in Whitt [20],  $M_1$  convergence for monotone functions is equivalent to pointwise convergence in a dense subset of points plus convergence at the endopoints, we conclude that  $d_{M_1}(\Phi_2^{(u)}(\eta_n), \Phi_2^{(u)}(\eta)) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $\Phi^{(u)}$  is continuous at  $\eta$ .  $\square$

**3.2. Main theorem.** Let  $(X_n)$  be a strictly stationary sequence of random variables, regularly varying with index  $\alpha \in (0, 1)$ . The theorem below gives conditions under which the joint partial sum and maxima process  $L_n(\cdot)$  satisfies a functional limit theorem with the limit  $L(\cdot) = (V(\cdot), W(\cdot))$ , where  $V(\cdot)$  is an  $\alpha$ -stable Lévy process and  $W(\cdot)$  is an extremal process.

The distribution of a Lévy process  $V(\cdot)$  is characterized by its characteristic triple, that is, the characteristic triple of the infinitely divisible distribution of  $V(1)$ . The characteristic function of  $V(1)$  and the characteristic triple  $(a, \nu', b)$  are related in the following way:

$$\mathbb{E}[e^{izV(1)}] = \exp\left(-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} (e^{izx} - 1 - izx1_{[-1,1]}(x)) \nu'(dx)\right)$$

for  $z \in \mathbb{R}$ . Here  $a \geq 0$ ,  $b \in \mathbb{R}$  are constants, and  $\nu'$  is a measure on  $\mathbb{R}$  satisfying

$$\nu'(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1) \nu'(dx) < \infty.$$

For a textbook treatment of Lévy processes we refer to Sato [17]. The distribution of an extremal process  $W(\cdot)$  is characterized by its exponent measure  $\nu''$  in the following way:

$$P(W(t) \leq x) = e^{-t\nu''(x, \infty)}$$

for  $t > 0$  and  $x > 0$ , where  $\nu''$  is a measure on  $(0, \infty)$  satisfying  $\nu''(\delta, \infty) < \infty$  for any  $\delta > 0$  (see Resnick [16], page 161).

The description of the characteristic triple of  $V(\cdot)$  and the exponent measure of  $W(\cdot)$  in the limit process will be in terms of the measures  $\nu'$  and  $\nu''$  on  $\mathbb{R}$  defined by

$$\nu'(dx) = (c_+ 1_{(0, \infty)}(x) + c_- 1_{(-\infty, 0)}(x)) \theta \alpha |x|^{-\alpha-1} dx$$

and

$$\nu''(dx) = r \theta \alpha x^{-\alpha-1} 1_{(0, \infty)}(x) dx,$$

where

$$c_+ = E \left[ \left( \sum_j \eta_{1j} \right)^\alpha 1_{\{\sum_j \eta_{1j} > 0\}} \right], \quad c_- = E \left[ \left( - \sum_j \eta_{1j} \right)^\alpha 1_{\{\sum_j \eta_{1j} < 0\}} \right]$$

and

$$r = E \left( \bigvee_j \eta_{1j} \vee 0 \right)^\alpha,$$

with  $(\eta_{1j})_j$  as defined in (2.7).

**Theorem 3.3.** *Let  $(X_n)$  be a strictly stationary sequence of random variables, jointly regularly varying with index  $\alpha \in (0, 1)$ , and of which the tail process  $(Y_i)_{i \in \mathbb{Z}}$  almost surely has no two values of the opposite sign. Suppose that Conditions 2.1 and 2.2 hold. Then the stochastic process*

$$L_n(t) = \left( \sum_{k=1}^{\lfloor nt \rfloor} \frac{X_k}{a_n}, \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} \right) \quad t \in [0, 1],$$

satisfies

$$L_n(\cdot) \xrightarrow{d} L(\cdot), \quad n \rightarrow \infty,$$

in  $D([0, 1], \mathbb{R}^2)$  endowed with the weak  $M_1$  topology, where  $L(\cdot) = (V(\cdot), W(\cdot))$ ,  $V(\cdot)$  is an  $\alpha$ -stable Lévy process with characteristic triple  $(0, \nu', c)$  where  $c = (c_+ - c_-) \theta \alpha / (1 - \alpha)$ , and  $W(\cdot)$  is an extremal process with exponent measure  $\nu''$ .

*Proof.* Take an arbitrary  $u > 0$ , and consider

$$\Phi^{(u)}(N_n)(\cdot) = \left( \sum_{i/n \leq \cdot} \frac{X_i}{a_n} 1_{\{\frac{|X_i|}{a_n} > u\}}, \bigvee_{i/n \leq \cdot} \frac{X_i}{a_n} \vee 0 \right).$$

From Lemma 3.1 and Lemma 3.2 we know that  $\Phi^{(u)}$  is continuous on the set  $\Lambda$  and this set almost surely contains the limiting point process  $N$  from (2.7). Hence an

application of the continuous mapping theorem yields  $\Phi^{(u)}(N_n)(\cdot) \xrightarrow{d} \Phi^{(u)}(N)(\cdot)$  in  $D([0, 1], \mathbb{R}^2)$  under the weak  $M_1$  topology, i.e.

$$L_n^{(u)} := \left( \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} 1_{\left\{ \frac{|X_i|}{a_n} > u \right\}}, \bigvee_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \vee 0 \right) \xrightarrow{d} L^{(u)} := \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{|P_i \eta_{ij}| > u\}}, \bigvee_{T_i \leq \cdot} \bigvee_j P_i \eta_{ij} \vee 0 \right). \quad (3.3)$$

Observe that the functions  $h_1, h_2: D([0, 1], \mathbb{R}^2) \rightarrow D([0, 1], \mathbb{R})$ , given by  $h_1(x) = x_1, h_2(x) = x_2$  for  $x = (x_1, x_2) \in D([0, 1], \mathbb{R}^2)$ , are continuous when  $D([0, 1], \mathbb{R}^2)$  and  $D([0, 1], \mathbb{R})$  are endowed with the weak  $M_1$  topology and the standard  $M_1$  topology, respectively. Hence from (3.3) by an application of the continuous mapping theorem we obtain

$$\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} 1_{\left\{ \frac{|X_i|}{a_n} > u \right\}} \xrightarrow{d} \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{|P_i \eta_{ij}| > u\}} \quad (3.4)$$

and

$$\bigvee_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \vee 0 \xrightarrow{d} \bigvee_{T_i \leq \cdot} \bigvee_j P_i \eta_{ij} \vee 0. \quad (3.5)$$

Let  $U_i = \sum_j \eta_{ij}$ ,  $i = 1, 2, \dots$ . For  $\alpha < 1$  it holds that

$$\mathbb{E}|U_1|^\alpha \leq \mathbb{E} \left( \sum_j |\eta_{1j}| \right)^\alpha < \infty \quad (3.6)$$

(see Davis and Hsing [8] and Mikosch and Wintenberger [14]). It is straightforward to check that  $P_i|U_i|$ ,  $i = 1, 2, \dots$ , are the points of a Poisson process with intensity measure  $\theta \mathbb{E}|U_1|^\alpha \alpha x^{-\alpha-1} dx$  for  $x > 0$  (see Proposition 5.2 and Proposition 5.3 in Resnick [16]). These points are summable (see the proof of Theorem 3.1 in Davis and Hsing [8]), and therefore for all  $t \in [0, 1]$

$$\sum_{T_i \leq t} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > u\}} \rightarrow \sum_{T_i \leq t} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > 0\}}$$

and

$$\sum_{T_i \leq t} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < -u\}} \rightarrow \sum_{T_i \leq t} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < 0\}}$$

almost surely as  $u \rightarrow 0$ . Since the processes  $\sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > u\}}$  are monotone for each  $u > 0$ , by Corollary 12.5.1 in Whitt [20] pointwise convergence implies convergence in the standard  $M_1$  topology, yielding

$$d_{M_1} \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > u\}}, \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > 0\}} \right) \rightarrow 0 \quad (3.7)$$

almost surely as  $u \rightarrow 0$ . Similarly we obtain

$$d_{M_1} \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < -u\}}, \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < 0\}} \right) \rightarrow 0 \quad (3.8)$$

almost surely as  $u \rightarrow 0$ . The tail process  $(Y_i)$  by assumption almost surely has no two values of the opposite sign, and hence the same property holds for the process  $(\sum_j \eta_{ij})_i$ . Therefore the limiting processes  $\sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > 0\}}$  and



$\sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < 0\}}$  have no common discontinuity point, and hence by Corollary 12.7.1. in Whitt [20] from (3.7) and (3.8) we obtain

$$d_{M_1} \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > u\}} + \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < -u\}}, \right. \\ \left. \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} > 0\}} + \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{P_i \eta_{ij} < 0\}} \right) \rightarrow 0,$$

i.e.

$$d_{M_1} \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} 1_{\{|P_i \eta_{ij}| > u\}}, \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij} \right) \rightarrow 0 \quad (3.9)$$

almost surely as  $u \rightarrow 0$ .

Let

$$L(\cdot) = \left( \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij}, \bigvee_{T_i \leq \cdot} \bigvee_j P_i \eta_{ij} \vee 0 \right).$$

Recalling the definition of the metric  $d_p$  in (2.8), from (3.9) we obtain

$$d_p(L^{(u)}(\cdot), L(\cdot)) \rightarrow 0$$

almost surely as  $u \rightarrow 0$ . Since almost sure convergence implies weak convergence, we have, as  $u \rightarrow 0$ ,

$$L^{(u)}(\cdot) \xrightarrow{d} L(\cdot) \quad (3.10)$$

in  $D([0, 1], \mathbb{R}^2)$  endowed with the weak  $M_1$  topology.

By Proposition 5.2 and Proposition 5.3 in Resnick [16], the process

$$\sum_i \delta_{(T_i, \sum_j P_i \eta_{ij})}$$

is a Poisson process with intensity measure  $Leb \times \nu'$ . Similarly, the process

$$\sum_i \delta_{(T_i, \bigvee_j P_i \eta_{ij} \vee 0)}$$

is an Poisson process with intensity measure  $Leb \times \nu''$ . By the Itô representation of the Lévy process (see Resnick [16], pages 150–153) and Theorem 14.3 in Sato [17],

$$V(\cdot) = \sum_{T_i \leq \cdot} \sum_j P_i \eta_{ij}$$

is an  $\alpha$ -stable Lévy process with characteristic triple  $(0, \nu', (c_+ - c_-)\theta\alpha/(1 - \alpha))$ . Also

$$W(\cdot) = \sum_{T_i \leq \cdot} \bigvee_j P_i \eta_{ij} \vee 0$$

is an extremal process with exponent measure  $\nu''$  (see Resnick [16], page 161).

If we show that

$$\lim_{u \rightarrow 0} \limsup_{n \rightarrow \infty} P(d_p(L_n(\cdot), L_n^{(u)}(\cdot)) > \epsilon) = 0$$

for any  $\epsilon > 0$ , from (3.3) and (3.10) by a variant of Slutsky's theorem (see Theorem 3.5 in Resnick [16]) it will follow that  $L_n(\cdot) \xrightarrow{d} L(\cdot)$  as  $n \rightarrow \infty$ , in  $D([0, 1], \mathbb{R}^2)$  with the weak  $M_1$  topology.

Since the metric  $d_p$  on  $D([0, 1], \mathbb{R}^2)$  is bounded above by the uniform metric on  $D([0, 1], \mathbb{R}^2)$  (see Theorem 12.10.3 in Whitt [20]), it suffices to show that

$$\lim_{u \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|L_n(t) - L_n^{(u)}(t)\| > \epsilon \right) = 0.$$

Using stationarity, Markov's inequality and the fact that

$$\left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} - \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} \vee 0 \right| \leq \frac{|X_1|}{a_n} 1_{\{X_1 < 0\}},$$

we get the bound

$$\begin{aligned} & \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|L_n(t) - L_n^{(u)}(t)\| > \epsilon \right) \\ &= \mathbb{P} \left( \sup_{0 \leq t \leq 1} \max \left\{ \left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} 1_{\left\{ \frac{|X_i|}{a_n} \leq u \right\}} \right|, \frac{|X_1|}{a_n} 1_{\{X_1 < 0\}} \right\} > \epsilon \right) \\ &\leq \mathbb{P} \left( \sum_{i=1}^n \frac{|X_i|}{a_n} 1_{\left\{ \frac{|X_i|}{a_n} \leq u \right\}} > \epsilon \right) + \mathbb{P} \left( \frac{|X_1|}{a_n} 1_{\{X_1 < 0\}} > \epsilon \right) \\ &\leq \epsilon^{-1} n \mathbb{E} \left( \frac{|X_1|}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq u \right\}} \right) + \mathbb{P} \left( \frac{|X_1|}{a_n} 1_{\{X_1 < 0\}} > \epsilon \right). \end{aligned} \quad (3.11)$$

For the first term on the right-hand side of (3.11) we have

$$n \mathbb{E} \left( \frac{|X_1|}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq u \right\}} \right) = u \cdot n \mathbb{P}(|X_1| > a_n) \cdot \frac{\mathbb{P}(|X_1| > ua_n)}{\mathbb{P}(|X_1| > a_n)} \cdot \frac{\mathbb{E}(|X_1| 1_{\{|X_1| \leq ua_n\}})}{ua_n \mathbb{P}(|X_1| > ua_n)}.$$

Since  $X_1$  is a regularly varying random variable with index  $\alpha$ , it follows immediately that

$$\frac{\mathbb{P}(|X_1| > ua_n)}{\mathbb{P}(|X_1| > a_n)} \rightarrow u^{-\alpha}$$

as  $n \rightarrow \infty$ . By Karamata's theorem

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(|X_1| 1_{\{|X_1| \leq ua_n\}})}{ua_n \mathbb{P}(|X_1| > ua_n)} = \frac{\alpha}{1 - \alpha}.$$

Therefore, taking into account relation (1.1), we get

$$n \mathbb{E} \left( \frac{|X_1|}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq u \right\}} \right) \rightarrow u^{1-\alpha} \frac{\alpha}{1 - \alpha}$$

as  $n \rightarrow \infty$ . Observe that

$$\frac{|X_1|}{a_n} 1_{\{X_1 < 0\}} \rightarrow 0$$

almost surely as  $n \rightarrow \infty$ , and thus

$$\mathbb{P} \left( \frac{|X_1|}{a_n} 1_{\{X_1 < 0\}} > \epsilon \right) \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore from (3.11) we obtain

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|L_n(t) - L_n^{(u)}(t)\| > \epsilon \right) \leq \epsilon^{-1} u^{1-\alpha} \frac{\alpha}{1 - \alpha}.$$

Letting  $u \rightarrow 0$ , since  $1 - \alpha > 0$ , we finally obtain

$$\lim_{u \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} \|L_n(t) - L_n^{(u)}(t)\| > \epsilon \right) = 0,$$

and this concludes the proof.  $\square$

**Remark 3.4.** The weak  $M_1$  convergence in Theorem 3.3 in general can not be replaced by the standard  $M_1$  convergence. This is shown in Example 4.1.

The problem in our proof if we consider the standard  $M_1$  topology is Lemma 3.2, which in this case does not hold. To see this, fix  $u > 0$  and define

$$\eta_n = \delta_{(\frac{1}{2} - \frac{1}{n}, \frac{u}{2})} + \delta_{(\frac{1}{2}, 2u)} \quad \text{for } n \geq 3.$$

Then  $\eta_n \xrightarrow{v} \eta$ , where

$$\eta = \delta_{(\frac{1}{2}, \frac{u}{2})} + \delta_{(\frac{1}{2}, 2u)} \in \Lambda.$$

It is easy to compute

$$\Phi_1^{(u)}(\eta_n)(t) = 2u 1_{[\frac{1}{2}, 1]}(t) \quad \text{and} \quad \Phi_2^{(u)}(\eta_n)(t) = \frac{u}{2} 1_{[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]}(t) + 2u 1_{[\frac{1}{2}, 1]}(t).$$

Then

$$y_n(t) := \Phi_2^{(u)}(\eta_n)(t) - \Phi_1^{(u)}(\eta_n)(t) = \frac{u}{2} 1_{[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]}(t), \quad t \in [0, 1],$$

and similarly

$$y(t) := \Phi_2^{(u)}(\eta)(t) - \Phi_1^{(u)}(\eta)(t) = 0, \quad t \in [0, 1].$$

Hence  $d_{M_1}(y_n, y) \geq \frac{u}{2}$  for all  $n \geq 3$ , which means that  $d_{M_1}(y_n, y)$  does not converge to zero as  $n \rightarrow \infty$ . Since

$$d_{M_1}(y_n, y) \leq d_{M_1}(\Phi^{(u)}(\eta_n), \Phi^{(u)}(\eta))$$

(see Theorem 12.7.1 in Whitt [20]), we conclude that  $d_{M_1}(\Phi^{(u)}(\eta_n), \Phi^{(u)}(\eta))$  does not converge to zero. Therefore the functional  $\Phi^{(u)}$  is not continuous at  $\eta$  with respect to the standard  $M_1$  topology.

**Remark 3.5.** Since for  $\alpha \in (0, 2)$  the stochastic processes  $V_n(\cdot)$  and  $W_n(\cdot)$  converge (separately) in the space  $D([0, 1], \mathbb{R})$  equipped with the standard  $M_1$  topology, it is naturally to expect that Theorem 3.3 holds also for  $\alpha \in [1, 2)$ . With the methods used in the proof of the theorem we have not been able to prove this conjecture. This is due to the fact that relation (3.6), which holds for  $\alpha < 1$ , may fail for  $\alpha \geq 1$  (see Mikosch and Wintenberger [14]).

#### 4. EXAMPLES

Various classes of stationary sequences are covered by our main theorem, such as squared GARCH processes, moving averages, moving maxima and ARMAX processes (see Basrak et al. [4] and Krizmanić [11]). Here we present in detail only moving maxima processes, for which we show that Theorem 3.3 fails to hold under the standard  $M_1$  topology on  $D[0, 1], \mathbb{R}^2$ .

**Example 4.1.** (Moving maxima). Let  $(Z_n)_{n \in \mathbb{Z}}$  be a sequence of i.i.d. Fréchet random variables with shape parameter  $\alpha \in (0, 1)$ , i.e.  $\mathbb{P}(Z_n \leq x) = e^{-x^{-\alpha}}$  for  $x > 0$ . Hence  $Z_n$  is regularly varying with index  $\alpha$ . Take a sequence of positive real numbers  $(a_n)$  such that  $n\mathbb{P}(Z_1 > a_n) \rightarrow 1$  as  $n \rightarrow \infty$ . Consider the finite order moving maxima

$$X_n = \max_{i=0, \dots, m} \{c_i Z_{n-i}\}, \quad n \in \mathbb{Z},$$

where  $m \in \mathbb{N}$  and  $c_0, \dots, c_m$  are nonnegative constants such that at least  $c_0$  and  $c_m$  are not equal to zero. Then the random process  $(X_n)$  is jointly regularly varying with index  $\alpha$  (see Example 2.1.12 in Tafró [19]). Since the sequence  $(X_n)$  is  $m$ -dependent, it follows immediately that Conditions 2.1 and 2.2 hold (see Example 5.1 in Krizmanić [11]).

Therefore  $(X_n)$  satisfies all the conditions of Theorem 3.3, and the corresponding stochastic processes  $L_n(\cdot)$  converge in distribution in  $D([0, 1], \mathbb{R}^2)$  under the weak  $M_1$  topology.

Next we show that  $L_n(\cdot)$  does not converge in distribution under the standard  $M_1$  topology on  $D([0, 1], \mathbb{R}^2)$ . This shows that the weak  $M_1$  topology in Theorem 3.3 in general can not be replaced by the standard  $M_1$  topology. In showing this we use, with appropriate modifications, a combination of arguments used by Basrak and Krizmanić [3] in their Example 4.1 and Avram and Taqqu [2] in their Theorem 1 (see also Example 5.1 in Krizmanić [13]).

For simplicity take  $m = 1$  and  $c_0 = c_1 = 1$ . We have  $X_n = Z_n \vee Z_{n-1}$  and  $L_n(t) = (V_n(t), W_n(t))$ , where

$$V_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \frac{X_j}{a_n} \quad \text{and} \quad W_n(t) = \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{X_j}{a_n}.$$

Let

$$G_n(t) := V_n(t) - 2W_n(t), \quad t \in [0, 1].$$

The first step is to show that  $G_n(\cdot)$  does not converge in distribution in  $D([0, 1], \mathbb{R})$  endowed with the (standard)  $M_1$  topology. For this, according to Skorohod [18] (see also Proposition 2 in Avram and Taqqu [2]), it suffices to show that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}(\omega_\delta(G_n(\cdot)) > \epsilon) > 0 \quad (4.1)$$

for some  $\epsilon > 0$ , where

$$\omega_\delta(x) = \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 \leq t_2 - t_1 \leq \delta}} M(x(t_1), x(t), x(t_2))$$

( $x \in D([0, 1], \mathbb{R})$ ,  $\delta > 0$ ) and

$$M(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\ \min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise,} \end{cases}$$

Note that  $M(x_1, x_2, x_3)$  is the distance from  $x_2$  to  $[x_1, x_3]$ , and  $\omega_\delta(x)$  is the  $M_1$  oscillation of  $x$ .

Let  $i' = i'(n)$  be the index at which  $\max_{1 \leq i \leq n-1} Z_i$  is obtained. Fix  $\epsilon > 0$  and introduce the events

$$A_{n,\epsilon} = \{Z_{i'} > \epsilon a_n\} = \left\{ \max_{1 \leq i \leq n-1} Z_i > \epsilon a_n \right\}$$

and

$$B_{n,\epsilon} = \{Z_{i'} > \epsilon a_n \text{ and } \exists l \neq 0, -i' \leq l \leq 1, \text{ such that } Z_{i'+l} > \epsilon a_n/4\}.$$

Using the facts that  $(Z_i)$  is an i.i.d. sequence and  $n\mathbb{P}(Z_1 > ca_n) \rightarrow c^{-\alpha}$  as  $n \rightarrow \infty$  for  $c > 0$  (which follows from the regular variation property of  $Z_1$ ) we get

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_{n,\epsilon}) = 1 - e^{-\epsilon^{-\alpha}} \quad (4.2)$$

and

$$\limsup_{n \rightarrow \infty} P(B_{n,\epsilon}) \leq \frac{\epsilon^{-2\alpha}}{4^{-\alpha}} \quad (4.3)$$

(see Example 5.1 in Krizmanić [11]).

On the event  $A_{n,\epsilon} \setminus B_{n,\epsilon}$  one has  $Z_{i'} > \epsilon a_n$  and  $Z_{i'+l} \leq \epsilon a_n/4$  for every  $l \neq 0$ ,  $-i' \leq l \leq 1$ , so that

$$W_n\left(\frac{i'}{n}\right) = W_n\left(\frac{i'+1}{n}\right) = \frac{Z_{i'}}{a_n} > \epsilon \quad \text{and} \quad W_n\left(\frac{i'-1}{n}\right) = \bigvee_{j=0}^{i'-1} \frac{Z_j}{a_n} \leq \frac{\epsilon}{4}.$$

Therefore after standard calculations we obtain

$$\left| G_n\left(\frac{i'}{n}\right) - G_n\left(\frac{i'-1}{n}\right) \right| = \left| -\frac{Z_{i'}}{a_n} + 2W_n\left(\frac{i'-1}{n}\right) \right| > \frac{\epsilon}{2} \quad (4.4)$$

and

$$\left| G_n\left(\frac{i'+1}{n}\right) - G_n\left(\frac{i'}{n}\right) \right| = \frac{Z_{i'}}{a_n} > \epsilon. \quad (4.5)$$

On the set  $A_{n,\epsilon} \setminus B_{n,\epsilon}$  it also holds that

$$G_n\left(\frac{i'}{n}\right) \notin \left[ G_n\left(\frac{i'-1}{n}\right), G_n\left(\frac{i'+1}{n}\right) \right],$$

which implies that

$$\begin{aligned} & M\left(G_n\left(\frac{i'-1}{n}\right), G_n\left(\frac{i'}{n}\right), G_n\left(\frac{i'+1}{n}\right)\right) \\ &= \min \left\{ \left| G_n\left(\frac{i'}{n}\right) - G_n\left(\frac{i'-1}{n}\right) \right|, \left| G_n\left(\frac{i'+1}{n}\right) - G_n\left(\frac{i'}{n}\right) \right| \right\}. \end{aligned}$$

Taking into account (4.4) and (4.5) we obtain

$$\begin{aligned} \omega_{2/n}(G_n(\cdot)) &= \sup_{\substack{t_1 \leq t \leq t_2 \\ 0 \leq t_2 - t_1 \leq 2/n}} M(G_n(t_1), G_n(t), G_n(t_2)) \\ &\geq M\left(G_n\left(\frac{i'-1}{n}\right), G_n\left(\frac{i'}{n}\right), G_n\left(\frac{i'+1}{n}\right)\right) > \frac{\epsilon}{2} \end{aligned}$$

on the event  $A_{n,\epsilon} \setminus B_{n,\epsilon}$ . Therefore, since  $\omega_\delta(\cdot)$  is nondecreasing in  $\delta$ , it holds that

$$\begin{aligned} \liminf_{n \rightarrow \infty} P(A_{n,\epsilon} \setminus B_{n,\epsilon}) &\leq \liminf_{n \rightarrow \infty} P(\omega_{2/n}(G_n(\cdot)) > \epsilon/2) \\ &\leq \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P(\omega_\delta(G_n(\cdot)) > \epsilon/2). \end{aligned} \quad (4.6)$$

Note that  $x^{2\alpha}(1 - e^{-x^{-\alpha}})$  tends to infinity as  $x \rightarrow \infty$ , and therefore we can find  $\epsilon > 0$  such that  $\epsilon^{2\alpha}(1 - e^{-\epsilon^{-\alpha}}) > 4^\alpha$ , i.e.

$$1 - e^{-\epsilon^{-\alpha}} > \frac{4^\alpha}{\epsilon^{2\alpha}}.$$

For this  $\epsilon$ , by relations (4.2) and (4.3), it holds that

$$\lim_{n \rightarrow \infty} P(A_{n,\epsilon}) > \limsup_{n \rightarrow \infty} P(B_{n,\epsilon}),$$

i.e.

$$\liminf_{n \rightarrow \infty} P(A_{n,\epsilon} \setminus B_{n,\epsilon}) \geq \lim_{n \rightarrow \infty} P(A_{n,\epsilon}) - \limsup_{n \rightarrow \infty} P(B_{n,\epsilon}) > 0.$$

Thus by (4.6) we obtain

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P(\omega_\delta(G_n(\cdot)) > \epsilon/2) > 0$$

and (4.1) holds, i.e.  $G_n(\cdot)$  does not converge in distribution in  $D([0, 1], \mathbb{R})$  endowed with the (standard)  $M_1$  topology.

If  $L_n(\cdot)$  would converge in distribution to some  $L(\cdot) = (V(\cdot), W(\cdot))$  in the standard  $M_1$  topology on  $D([0, 1], \mathbb{R}^2)$ , then using the fact that linear combinations of the coordinates are continuous in the same topology (see Theorem 12.7.1 and Theorem 12.7.2 in Whitt [20]) and the continuous mapping theorem, we would obtain that  $G_n(\cdot) = V_n(\cdot) - 2W_n(\cdot)$  converges to  $V(\cdot) - 2W(\cdot)$  in  $D([0, 1], \mathbb{R})$  endowed with the standard  $M_1$  topology, which is impossible, as is shown above.

#### ACKNOWLEDGEMENTS

This work has been supported in part by Croatian Science Foundation under the project 3526.

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